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A short history of fractal-Cantorian space-time

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ABSTRACT

The article attempts to give a short historical overview of the discovery of fractal-Cantorian space-time starting from the 17th century up to the present. In the last 25 years a great number of scientists worked on fractal space-time notably Garnet Ord in Canada, Laurent Nottale in France and Mohamed El Naschie in England who gave an exact mathematical procedure for the derivation of the dimensionality and curvature of fractal space-time fuzzy manifold.

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1. Historical overview of ideas leading to fractal space-time

The idea of hierarchy and fractal-like self-similarity in science started presumably first in cosmology before moving to the realm of quantum and particle physics [1]. It is possible that the English clergyman T. Right was the first to entertain such ideas (Fig. 1). Later on the idea reappeared in the work of the Swedish scientist Emanuel Swedenborg (1688–1772) and then much later and in a more mathematical fashion in the work of another Swedish astrophysicist C. Charlier (1862–1934) (Fig. 2).

In 1983, the English-Canadian physicist Garnet Ord wrote a seminal paper [2] and coined the phrase Fractal Space-time. Ord set out to take the mystery out of analytical continuation. We should recall that analytical continuation is what converts an ordinary diffusion equation into a Schrödinger equation and a telegraph equation into a Dirac equation. Analytical continuation is thus a short cut quantization. However, what really happened is inexplicable. Ord showed using his own (invented) quantum calculus, that analytical continuation which consists of replacing ordinary time t by imaginary time it where $i = \sqrt{-1}$ is not needed if we work in a fractal-like setting, i.e. a fractal space-time. Although rather belated, Ord's work has in the mean time gained a wider acceptance and was published for instance in Physic Review [3]. Therefore, one is hopeful that his message has found wider understanding. It is the transfinite geometry and not quantization which produces the equations of quantum mechanics. Quantization is just a very convenient way to reach the same result fast, but understanding suffers in the process of a formal analytical continuation.

Similar work, but not identical, was carried out by the French cosmologist Laurent Nottale, fifteen years ago. Nottale wrote a book entitled: Fractal Space-time and microphysics, published by World Scientific in 1993 [4]. Nottale connected scaling and Einstein's relativity to what is now called scale relativity theory.

Then came the next quantum jump, around 1990, when M.S. El Naschie who was originally working on elastic and fluid turbulence began to work on his Cantorian version of fractal space-time. He showed that the n-dimensional triadic Cantor set has the same Hausdorff dimension as the dimension of a random inverse golden mean Sierpinski space to the power n-1 [5].

In the year 1995 Nobel laureate Prof. Ilya Prigogine, Otto Rössler and M.S. El Naschie edited an important book entitled: Quantum mechanics, diffusion and chaotic fractals; Pergamon (Elsevier – ISBN 0 08 04227 3) (1995) [6] in which the basic principles of fractal space-time were spelled out. Sometime later El Naschie using the work of Prigogine on irreversibility showed that the arrow of time may be explained in a fractal space-time [7–16]. A few years later two of El Naschie's papers

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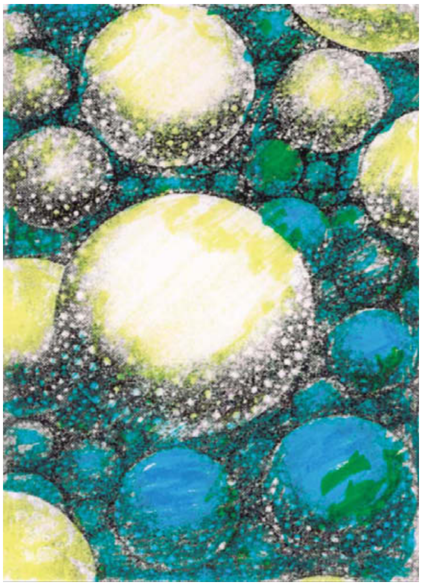


Fig. 1. A. vision of T. Right's cosmos as a form of sphere packing on all scales (Courtesy M.S. El Naschie).

Fig. 2. A vision of a fractal-like universe, with clusters of clusters ad infinitum as envisaged by the Swedish astronomer C. Charlier who lived between 1862 and 1934. This work was clearly influenced by the work of the Swedish astrophysicist A. Swedenborg (1688–1772) (Courtesy M.S. El Naschie).

on the subject were noted by Thompson essential science indicators as the most cited New frontier paper in physics and as Hot paper in engineering [20,21].

2. Concept of E-infinity theory

In the following years a basic conceptual framework of a new space-time theory with applications to high energy particle physics started to emerge [1–21]. The main idea of El Naschie's E-infinity theory is in fact a sweeping generalization of Einstein's general theory of relativity, namely introducing a new geometry for space-time which differs from the space-time of our sensual experience. Physical space-time is taken for granted to be smooth and flat Euclidean geometry. General relativity persuaded us that the Euclidean 3 + 1 dimensional space-time is only an approximation and that the true geometry of the universe in the large is in reality a four dimensional curved manifold. In E-infinity theory we take a similar step and argue that space-time at quantum scales is far from being the smooth, flat and passive space which we use in classical physics. In particular the geometry of chaotic dynamics, namely fractal geometry is reduced to its quintessence, i.e. Cantor sets and is employed directly in the geometrical description of the vacuum fluctuation of space-time (El Naschie MS. VAK, vakuum fluctuation and the mass spectrum of high energy particle physics. Chaos, Solitons & Fractals 2003;17: 797–807) [1,7,8,20,21].

As well known, special relativity fused time and space together, then came general relativity and introduced a curvature to space-time. Kaluza and later on Klein added one more dimension to the classical four in order to unify general relativity and electromagnetism. This extra fifth dimension was supposed to be rolled up in a tiny pipe with a radius of the order of magnitude of 10^{-33} cm. The dimensionality of space-time played a paramount role in the theoretical physics of unification and lead to the introduction of the 26 dimensions of string theory, the 10 dimensions of super string theory and finally the heterotic string theory with the dimensional hierarchy 4, 6, 10, 16 and 26 [1,7,20,21].

In E-infinity theory El Naschie admit formally infinite dimensional "real" space-time. This infinity is hierarchical in a strict mathematical way and he was able to show that E-infinity has finite number of dimensions when observed from a distance. At low resolution or equivalently at low energy the E-infinity Cantorian space-time appear as a four dimensional space-time manifold. The four dimensionality is a probabilistic statement, a so-called expectation value. The Hausdorff dimensions of this topologically four dimensional-like "pre" manifold is also a finite value equal to $4 + \phi^3 = 4.236067977$, where $\phi = \frac{\sqrt{5}-1}{2}$ with the remarkable Russian doll-like self-similar continued fraction representation (see Figs. 9 and 10)

$$4 + \phi^3 = 4 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \dots}}}$$

It is noteworthy that $4 + \phi^3$ is equal to the Hausdorff dimension of a Mauldin-Williams random triadic Cantor set namely $D = \frac{\sqrt{5}-1}{2} = \phi$ lifted to four dimensions $\left(\frac{1}{\phi}\right)^{4-1} = \left(\frac{1}{\phi}\right)^{3} = 4 + \phi^3$, (see Figs. 9 and 10) [8].

If we project the space-time of vacuum fluctuation on a Poincaré circle we will see a hyperbolic tessellation of this circle with predominantly Klein-curve-like geometry. This is an important part of El Naschie's thesis that actual quantum space-time strongly resembles the hyperbolic geometry of the ramified $\Gamma\left(\frac{3\pi}{7}\right)$ Klein-modular-curve. The limit set of the well known Möbius-Klein transformation of this space which may be represented using the Beltrami-Poincaré methods of hyperbolic geometry is an elementary random Cantor set (Fig. 3). Quantum space-time is conceived here as a hyperbolic fractal in which the low resolution 336 triangles major parts is the original Klein-curve whiles the high resolution part at the circular boundary may be considered as transfinite correction using infinitely many increasingly smaller triangles. This procedure is called compactification. The total equivalent number is nominally equal 336 + 3 = 339. The derivation of $\overline{\alpha}_0 = 137$ is explained in Fig. 11. The conceptual similarity between the tiling of Fig. 3 and Regge calculus should also be noted [17].

Deterministic chaos is well documented in the literature particularly in Hamiltonian systems in connection with the work of Henri Poincaré. This pioneering work was then extended by Kolmogorov and others and finally led to the realization by the French topologist René Thom that the stationary states of quantum mechanics could be modeled by the VAK (vague attractor of Kolmogorov, Fig. 4). It is a composition of periodic orbits and chaotic islands which may be regarded as an extension of the notion of an attractor to Hamiltonian systems. Remarkably the role of the stabilizing frictional forces which is totally absent from Hamiltonian systems and quantum mechanics is assumed here by the irrationality of the winding number. The more irrational the winding number is (The ratio of the resonance frequencies) the more stable is the periodic orbit. Since the golden mean is the most irrational number, it follows that the orbit with the golden mean as a winding number is the most stable. That is the reason behind the frequent appearance of the golden mean in El Naschie's work [20,21].

Thus it is understandable that the ground state of quantum vacuum fluctuation is modeled in E-infinity theory via the VAK and that the golden mean random Cantor set plays there a profound role. The connection between the VAK and the golden mean is summarized in KAM theorem (Kolmogorov, Arnold, Moser). One of the most important results of E-infinity theory is to reason that the question of stability of elementary particles is closely related to KAM theory, Arnold diffusion and the

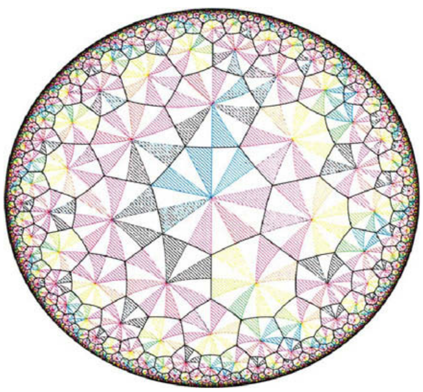


Fig. 3. The holographic boundary of fractal space-time tiling the plane using Klein's-modular-curve with 336 + 3 = 339 degrees of freedom in the Beltrami–Poincaré representation. E-infinity theory alleges that quantum gravity space–time is a hyperbolic fractal on a Klein-modular group akin to what is shown in this figure. The derivation of $\overline{\alpha}_0 = 137$ using the holographic boundary is shown in Fig. 11. Note the similarity to Regge triangulation [17].

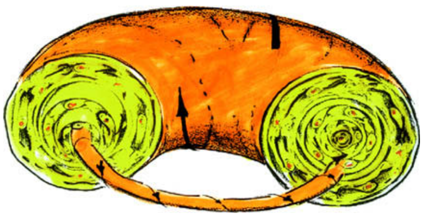


Fig. 4. The vague attractor of Kolmogorov (VAK) [7] which models the stationary states of quantum mechanics.

vague attractor of Kolmogorov. The most irrational number namely $\phi = \frac{\sqrt{5}-1}{2}$ is thus the secret of the stability of certain elementary particles. Vibration simulating particles which do not have a sufficient irrational winding number dissipate as fast as they are produced [1,20,21].

3. The holographic principle

The holographic principle first conjectured by the Dutch scientist, Nobel laureate Gerardus 'tHooft has been the subject of many recent fundamental contributions. El Naschie found the connection of E-infinity theory to the dimension of the original Klein-modular space Γ (7). While dim $\Gamma(7) = 336$, when compactified using the transfinite correction, the dimension becomes dim $\Gamma_C(7) = 336 + 16$ k, where $k = \phi^3(1-\phi^3) = 0.18033989$ and $\phi = \frac{\sqrt{5}-1}{2}$. Thus dim Γ_C (7) = 338.8854382 is the compactified holographic boundary given by the compactified holographic Klein-modular curve shown in Fig. 3. El Naschie used $\Gamma_C(7)$ to deduce $\overline{\alpha_0} = 137$ of the electromagnetic field (see Fig. 11) [9]. The eigenvalues like equation have a very simple interpretation: Dim E_8 E_8 = 496 represent all fundamental interactions. Thus it is equal to particle physics 339 symmetries plus the $R^{(4)} = 20$ of gravity plus $\overline{\alpha_0}$. From that we deduce $\overline{\alpha_0} = 496 - 339 - 20 = 137$.

4. Lie groups and Stein spaces hierarchy

El Naschie's Cantorian theory undergone an important transformation by the recent discovery of the exceptional Lie groups and Stein space hierarchy.

The exceptional Lie group hierarchy may be constructed in numerous ways but all of them include the exceptional Lie groups E_8, E_7 and E_6. One of the possible sums is presented graphically in Fig. 5 and gives the integral value 548, the transfinite corrected value is $548 + 4k_0 = 548.3281572$ where $k_0 = \phi^5(1-\phi^5) = 0.082039325$ [10].

The derivation of the inverse electromagnetic fine structure from the dimension of the exceptional Lie symmetry groups manifold E_8E_8 plays a significant role in developing E-infinity theory. The integral value of the inverse electromagnetic fine structure constant which was found is $\overline{\alpha_0} = 137$ while the transfinite corrected value is $\overline{\alpha_0} = (20)\left(\frac{1}{\phi^4}\right) = 137 + k_0 = 137.082039325$.

It was shown that the sum of Lie groups hierarchy 548 is four times the inverse electromagnetic constant, $548 = (4)(137) = (4)\overline{\alpha_0}$ while the transfinite value is $(4)(137 + k_0) = 548 + 4k_0$. This speaks clearly for the geometric topological origin and meaning of $\overline{\alpha_0} = 137$.

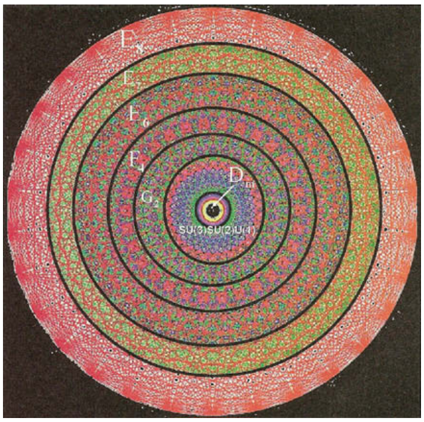


Fig. 5. The exceptional Lie group hierarchy (Courtesy M.S. El Naschie) with dimensions $E_8 = 248$, $E_7 = 133$, $E_6 = 78$, $E_5 = 45$, $E_4 = 24$, $E_3 = 11$, $E_2 = 6$, $E_1 = 3$. The sum is 548.

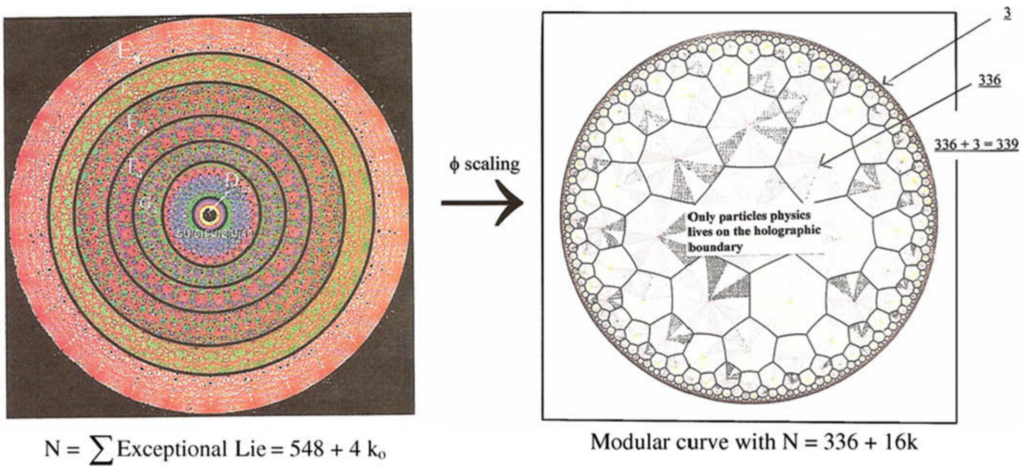


Fig. 6. Transfinite scaling transformation from exceptional Lie group hierarchy to the holographic boundary represented by the compactified Klein-modular curve $\Gamma(7)$.

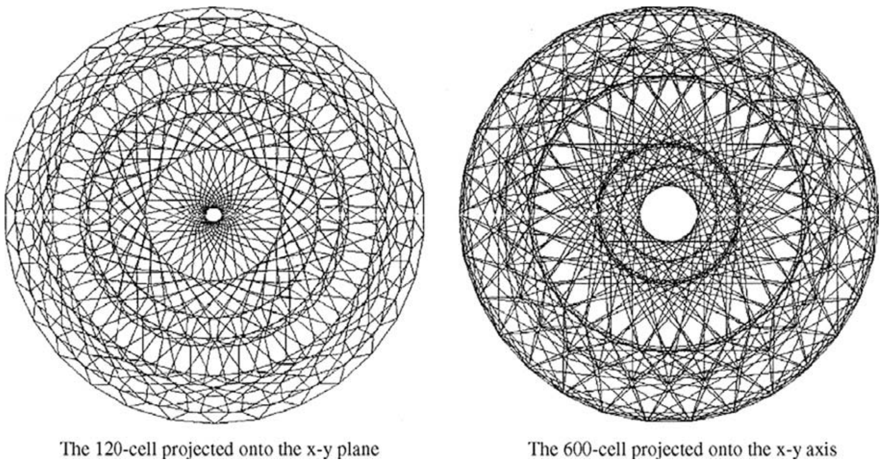


Fig. 7. Note that both Coxeter polytopes live in 4D and are dual to each other. By contrast Ji-Huan He's polytope lives in 10 dimensions and has a total number of elements equal to (69)(128) = 8832 times two. This is 9600 elements or states more than heterotic string theory with (63)(128) = 8064 states. Note further that 69 are the number of elementary particles in the standard model and that is the reason for many authors to think that there are still nine particles to be discovered to complete our so called standard model of elementary particles.

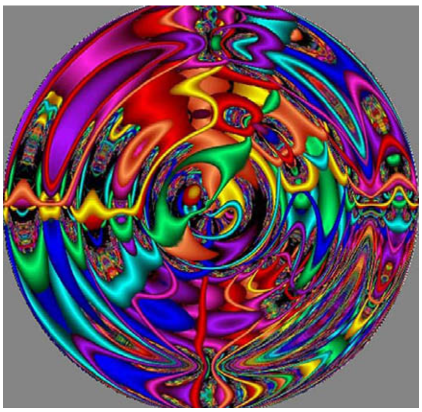


Fig. 8. The fuzzy Kähler manifold of E-infinity following G. Iovane.

Even more remarkable than that is the result found from adding all the dimensions of all two and three-Stein spaces. The result which the readers can easily verify is that adding the dimensions of those 17 spaces gives $686 = 5\overline{\alpha}_0 + 1$ where $5\overline{\alpha}_0 = 685$. The transfinite corrected value is $5\overline{\alpha}_0 = 685.4101967$[10].

Even more recently El Naschie announced an astonishing result connecting the 219 three dimensional crystallographic groups corresponding to the 17 Stein spaces. He found a quasi dimension of the sum of all these 219 groups equal to 8872 which is the exact integer number of the first level of massless states of heterotic string theory.

5. Coxter's 120-cell, Kähler and other manifolds as models of E-infinity

The regular Coxeter polytopes are well known examples of Coxeter's groups and play a fundamental role in Cantorian E-infinity theory. For instance, the four-dimensional polytopes called 24-, 120- and 600-cell were used in crisp and fuzzy

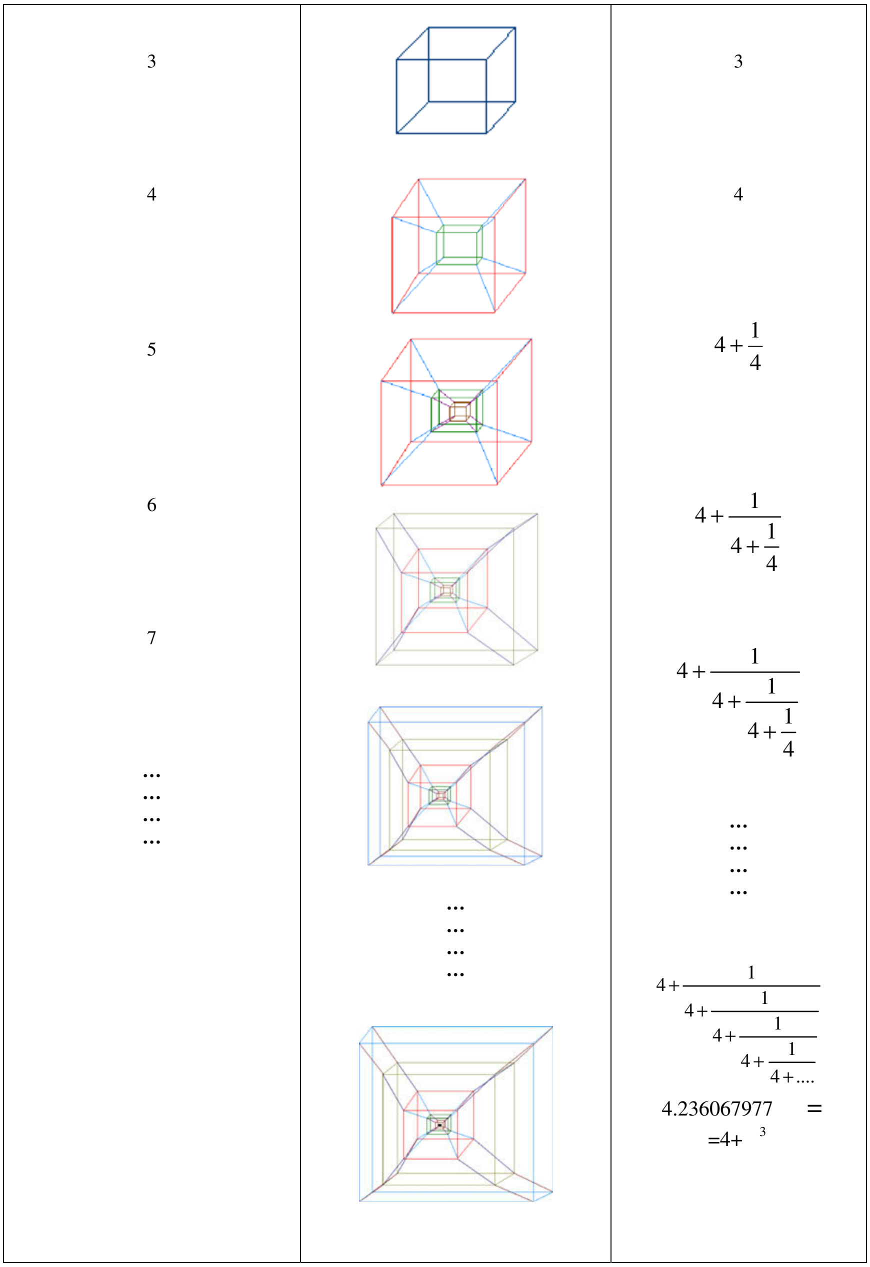


Fig. 9. Fractal dimensions of hyper cube starting from a three dimensional cube. A process of cube nesting leads at infinity to the average or expectation value of the Hausdorff dimension of fractal-Cantorian space-time $4 + \phi^3$ when $\phi = \frac{\sqrt{5}-1}{2}$.

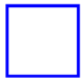
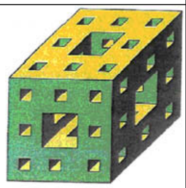
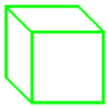
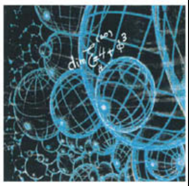
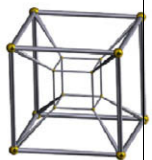
Type of fractal	Geometrical shape	Menger–Urysohn dimension	Hausdorff dimension	Correspondig random Hausdorff dimension	Embeddig dimension	Corresponding Euclidean shape	Remark
Cantor Set		0	*ln 2/ln3 = 0.630929753*	$\phi = 0.618033$	1	*Line*	The middle third of the line is removed and the iteration is repeated to obtain Cantor set. The final total length is zero.
Sierpinski gasket		2	*ln 3/ln2 = 1.584962501*	$\frac{1}{\phi} = 1.618033$	2	 *Square*	Hausdorff dimension of this fractal is the inverse of the Hausdorff dimension of the classical cantor set.
Menger sponge		3	D_{MS} = *ln20/ln3 = 2.7268*	$2 + \phi = 2.61803398$	3	 *Cube*	The COBE temperature of microwave background radiation is found to be T_c(COBE) = D_{MS}K = 2.726 K.
The 4 dimension random cantor set analogue of Menger sponge	 *An artist impression of $\varepsilon^{(\infty)}$ space-time*	4	$d_c^{(4)} = 4.236068$	$4 + \phi^3 = 4.23606797$	5	 *Hyper cube*	Note that E-infinity was not postulated but rather motivated by physical considerations. It was derived mathematically from first principles using set theory.

Fig. 10. Fractal–Hausdorff dimensions from E-infinity point of view. Notice that our classification and comparison with orderly classical Cantor set, Sierpinski gasket, Menger sponge and the 4 dimensional analogs of the Menger sponge and hypercube is almost complete. We just need the exact chaotic fractal shape of the fractal hypercube which is given here only as an artist impression (Courtesy M.S. El Naschie).

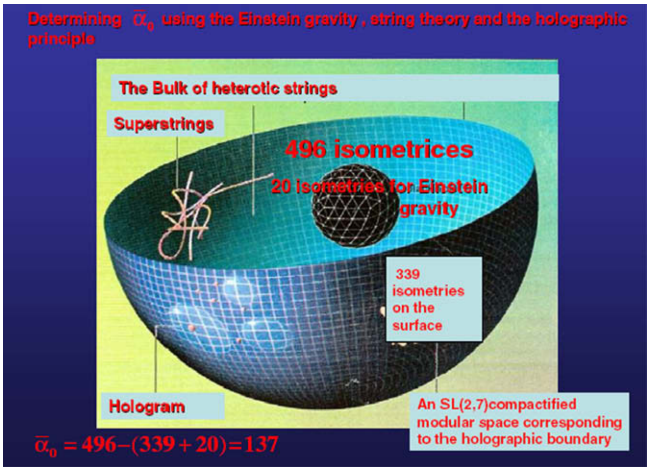


Fig. 11. Holographic principle applied to string theory. All fundamental interactions are represented by DimE_8E_8 = 496. This is equal to $339 + \bar{\alpha}_0 + R^{(4)}$. Since $R^{(4)} = 20$ for Einstein's relativity and particles physics lives only on the holographic boundary represented by 339 states it follows that $\bar{\alpha}_0 = 496 - [339 + 20] = 137$ as should be.

form in connection with E-infinity theory [see Figs. 7 and 8]. The number of elements of the 24-cell polytope is exactly equal to the roots number of E_8. A surprise was that there is a four-dimensional hyperbolic manifold M^4 which is build on a certain Coxeter 120-cell polytope with the Euler characteristic $\chi(M^4) = 26$. An important result is the possibility of viewing E-infinity Cantorian space-time as a transfinite extension of a four dimensional hyperbolic manifold M^4 with an Euler characteristic equal to the quantum gravity coupling constant $\overline{\alpha_{gs}} = 26 + k$ of E-infinity. E-infinity space-time may be regarded as fuzzy version of M^4 [11] (see Fig. 8).

Following the Gauss-Bonnet equation for the calculation of the volume of the higher dimensional hyperbolic manifolds, we obtain the remarkable result $2\mathrm{Vol}_4(M^4) = 684$, which is the dimension of Munroe's quasi exceptional Lie group E_{12} [12]. (See Fig. 8.)

The transfinite correction of this value is exactly the well known sum of two and three-Stein spaces 685.4101965 [13].

Another model of E-infinity theory is a class of fuzzy Kähler-like manifolds which can lead to few significant mathematical and physical results. The crisp Kähler K_3 manifold with the Euler characteristic $\chi = 24$ equal to the number of instantons plays an important role in super string theory. The fuzzy K(E-infinity) Kähler-like manifold (Fig. 6) on the other hand possesses the Euler characteristic $\chi = 26 + k = 26.18033986$. E-infinity itself could be regarded as a fuzzy Kähler manifold for a fuzzy 120-cell Coxeter polytope [14–16].

Finally few years ago El Naschie was able to show that the curvature of fractal- Cantorian space-time corresponding to the Hausdorff dimension $4 + \phi^3$ is exactly 26+k. Consequently the energy is proportional to $(26 + k)^2 = 685.4101967$ which is in turn the sum over all Stein space dimensions and the dimension of E_{12}[13]. This clearly is a major extension of the basic idea of Feynman's path integral [18] to exceptional Lie groups and Stein spaces as well as three dimensional crystallographic groups.

6. The work of Ambjorn, Loll and their school

A particularly interesting subject related to fractal-Cantorian space-time is the computer simulation research program initiated by Ambjorn, Loll and Jurkiewicz. The subject was considered in a recent Scientific American article and subsequent intensive discussion [17]. It is hoped that both approaches, the analytical and the simulation can benefit from each other and lead to an even more fundamental understanding of quantum gravity.

7. Conclusion

We have given here a very short historical account of the beginning of the theory of fractal space-time and its genesis to E-infinity theory and the summing over all Lie groups and Stein spaces approach. Many important contributions to the subject were omitted. Also we did not consider the stimulating early impulses which came from the work of R. Feynman as well as the discovery of the area-like quantum path by Abbot and Weiss [19] as well as the work of G. Parisi in Italy. Never the less we hope that we were still able to capture the most important ideas and concepts leading to a fascinating theory.

Acknowledgements

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