

Multi-dimensional Cantor Sets in Classical and Quantum Mechanics

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Abstract—We give two different descriptions of an abstract n -dimensional dynamical system. First we use a Sierpinski space setting and subsequently we use a statistical cellular space setting. The results of the analysis elucidate certain universal behaviour which was observed in a wide category of cellular automata. The results further show that in four dimensions the phase space dynamics is Peano-like and resembles an Anosov diffeomorphism of a compact manifold which is dense and quasi-ergodic. The fractal dimension in this case is $d_c^{(4)} = 3.981 \cong 4$ and we conjecture that fully developed turbulence is related to $d_c^{(5)} = 6.3$. The corresponding Shannon information entropy of the second analysis are $\mathcal{H}_s^{(4)} = 3.68$ and $\mathcal{H}_s^{(5)} = 6.12$. In the case of eight dimensional phase space both descriptions lead to almost identical numerical results. Possible implications of these theoretical results to physical spatio-temporal chaos and the reduction of complexity are discussed. In conclusion, the relevance of Cantor-like space-time for the Copenhagen interpretation of quantum mechanics and the connection to non-standard analysis and Boscovich covariance are touched upon.

1. THE GENERALIZED SIERPINSKI SPACE

In some recent publications [1, 2, 15], the Hausdorff capacity ($d_c^{(n)}$) of multidimensional Cantor sets was considered. There it was shown that the (n) -dimensional triadic Cantor set has the same Hausdorff dimension as the dimension of a Sierpinski space to the power $(n - 1)$. In the present work we give another interpretation of the same analysis, namely that triadic Cantor sets ($d_c = \log 2/\log 3$) can be generalized in n dimensional space if d_c is magnified by the inverse of a corresponding Buffon-like geometrical quotient [3, 4]. This factor ($\mathcal{L}^{(n)}$) is given by

$$\mathcal{L}^{(n)} = A_E^{(n)}(d_E^{(n)})/A_c^{(n)}(d_c^{(n)}) = 1/(d_c)^n = d_s^n$$

where

$$A_E^{(n)}(d_E^{(n)}) = 1 \quad \text{and} \quad A_c^{(n)}(d_c^{(n)}) = (d_c)^n$$

are the unit hyper-area of the Euclidean space ($d_E^{(n)}$) and the corresponding quasi-area of the Cantor space ($d_c^{(n)}$) respectively. Here a bracketed superscript refers to the dimension of the set and not to a power. Consequently, $1/\mathcal{L}^{(n)}$ may be viewed either as a geometrical, pseudo measure [3, 4] or alternatively as the dimension of a generalized Sierpinski space. Both ways one finds a simple series for the relevant dimensions (see Appendix 2)

$$d_c^{(n)} = (1/d_c)^{n-1} = d_s^{n-1}.$$

Evaluating this for the triadic Cantor set from $n = 1$ to $n = 8$, the first column of Table 1 is found. Note that at $n = 4$ we have $n \cong d_c^{(4)} \cong 3.98159$ and the set is thus almost “space filling” as previously pointed out [1, 2]. Note also that $d_c^{(2)} = 1.58496$ is identical to the fractal dimension found for a wide class of cellular automata reported in [18]. Next we will

Table 1

Dimension n	1 Capacity dimension $d_c^{(n)}$	2 Maxwell– Boltzmann $\delta_B = S_B/K_B$	3 Gibbs– Shannon $\delta_s = S_s/K_B$	4 Bose– Einstein $\delta_E = S_E/K_B$
2	$1.5849 \equiv \frac{\pi}{2}$	1.0986	0.69314	1.38629
3	$2.5121 < n$	2.3025	1.7351	2.70805
4	$3.98159 \equiv n$	3.85014	3.68148	4.025
5	$6.3106 \equiv 2\pi$	5.5053	6.1202	5.3471
	$> n$			
6	10.002	7.3790	10.3837	6.6745
7	15.853	9.3484	16.0277	8.00736
8	$25.126 \equiv 8\pi$	11.46	$25.101 \equiv d_c^{(8)}$	9.34487

seek to strengthen these results using the statistical method [5–7]. In particular we give a justification for the use of the triadic set as an organizing centre* as well as to the observation that all initial cellular automata leads to configurations with identical statistical behaviour [18].

2. “UNFOLDING” AND PARTITIONING OF THE “PHASE SPACE”

The basic concept of the present statistical approach is the following. Similar to lattice gases method and Ising model we unfold the n -dimensional Cantor set in two dimensions. This space which may be regarded as a configuration space or a phase space of an abstract dynamical set will be divided into lattice sites, one for each dimension. Each dimension is simultaneously regarded as a particle of a hypothetical gas. In the case of a micro quantum space, time will be regarded as spacialized. Finally we regard the various ways in which these n ‘particles’ may be distributed among the different $N = n$ ‘cells’ as an indirect representation of the ‘phase space’ dynamics of the set. Note that we have $N_{\max} = n$ in case of a maximum entropy. However for all other possible configurations we may have empty cells because of the assumption $n = N$. Note also that we are making no distinction between momenta and coordinates in our model exactly as in a Hamiltonian.

All that is needed now is an occupational rule for the ‘ n ’ particles which define a suitable ensemble for the lattice system. After some modification the statistical model chosen can be given *a posteriori* satisfactory interpretation.

3. THE M–B STATISTICS

In what follows an effective absolute entropy for our abstract dynamical set which is a modification of the classical Maxwell–Boltzmann statistics will be used [4–7]. In this case, the complexion is given combinatorically by [7, 8]

$$\omega_i = (n!)/(n_1!n_2! \dots n_n!); \sum n_i = n$$

where n_1, n_2, \dots are subsets of n . Assuming an equi-probable ensemble and disregarding any empty cell as well as any configuration which is formed by group symmetric transformation we can calculate a number for the total additive geometric complexion

*The triadic set may be thought of as a kind of a resultant of a Borel set.

$$\bar{\eta}_i = \sum_{i=1}^{i=P(n)} \bar{\eta}_i$$

where $P(n)$ is the classical unrestricted partition numbers. The total geometrical, non-additive entropy of the entire dynamics is then given simply by [7, 8]

$$S_B^{(n)} = S_B^{(n)}/K_B = \ln \left\{ \sum_{i=1}^{i=P(n)} [(n!)/(n_1!n_2! \dots n_n!)] \right\}$$

where K_B plays the role of a Boltzmann constant while $\bar{\eta}_i$ plays the role of the discrete and finite internal time necessary for the evolution of the system towards a steady state. The result of the calculations is given in the second column of Table 1. Note that the most important conclusions of [1, 2] with regard to the Hausdorff dimension ($d_c^{(n)}$) are reinforced by the behaviour of S_B . We see clearly that at $n = 4$ we have $S_B^{(4)} \cong d_c^{(n)} \cong d_c^{(4)} = n = 4$ and that for $n > 4$ we have a change in the sign of the codimension [1, 2]. Note that the partitioning number may be easily calculated using Hardy–Ramanujan approximation which gives excellent results. An alternative formulation as well as an approximate method for calculating $\bar{\eta}_i$ are given in Appendix 1.

4. INFORMATION ENTROPY

The generalization of the preceding modified physical entropy to that of the additive information entropy formula [8]:

$$\Delta S_s^j = \sum_{i=1}^{i=N} \mathcal{P}_i \ln 1/\mathcal{P}_i; \quad S_s^{(n)} = S_s^{(n)}/K_B = \sum_{j=1}^{j=P(n)} \Delta S_s^j$$

is reached by using the Stirling approximation, averaging over n and summing for all realizable configurations. Here \mathcal{P}_i is the probability of finding one particle in a cell. We apply this formula under the same previous assumptions and restrictions to see if it yields better results. This expectation is shown in column 3 of Table 1 to be fulfilled for large n ($n > 7$). Note that here, as before, we find for $n = 4$ that $S^{(4)} \leq d^{(n)} \cong n = 4$ while for $n = 5$ we have $S^{(5)} \cong d_c^{(5)} \cong 6.3 > n = 5$.

Another easier way to calculate $S^{(n)}$ and without using the Stirling approximation is of course by appealing directly to the average additive entropy

$$S^{(n)}/K_B = \left[\sum_{i=1}^{i=P(n)} \log_2 \bar{\eta}_i \right] / n \cong [(S_I^{(n)}/K_B) \ln 2] / \ln \varepsilon = I$$

where $\varepsilon = \sqrt[n]{\bar{n}}$ is the side length of the hyper-lattice covering our point set, \bar{n} is the occupational number, I is the information dimension and $\ln \varepsilon \cong \ln 2$. The results of these calculations are displayed in Table 2. We see clearly that $S_I^{(n)}$ behaves qualitatively very similarly to $S_s^{(n)}$ and $d_c^{(n)}$ and that for $n > 5$ we have $S_I^{(n)} \cong S_s^{(n)} \cong d_c^{(n)}$. It is evident that the results of the preceding calculations indicate the existence of a certain class of universal behaviour with dimensionality playing the key role.

5. QUANTUM ENSEMBLE

Now we apply statistics satisfying the symmetry restrictions of Bosons mechanics [6, 7] to the problem at hand. This is done in column 4 of Table 1 by using the Bose–Einstein ensemble* [4, 7]

*Fermi–Dirac statistics do not lead to the same present conclusions.

$$\zeta_E^{(n)} = [(n + N - 1)!]/[n!(N - 1)!].$$

The result for $n = 4$ is almost identical to that of the Hausdorff dimension ($d_c^{(n)}$). This proximity is a consequence of the relationship between Pascal triangle and the Sierpinski gasket. In particular and as shown in Appendix 2 the power series $d_c^{(n)}$ may be interpreted as the simple reciprocal value of that obtained using the multiplication rule of simultaneous events in a probability space [3–7]. Here there might be some interesting connections to the Feynman path integrals approach and his equation (2-35) (p. 38 of [19]). A systematic explanation of these relationships will be given elsewhere.

6. GEOMETRICAL INTERPRETATION

It would of course be a valuable addition for understanding the results of Table 1 if we could visualize these higher dimensional triadic sets geometrically. This inborn need of the human soul for geometrical intuitive interpretation is of course not possible for higher dimensions in general. However for two and three dimensions it is relatively easy to give an approximate but simple geometrical picture of how these sets come about. In two dimensions we can argue geometrically in the following manner. If we look at the Cartesian product of two triadic sets, then we see that they are triadic Cantorian along the perimetric and the diagonal directions but not in the two principal directions of the core. On the other hand the Sierpinski carpet represents in effect the reversed situation. Consequently a generalized Cartesian product averaging the capacity dimension of both sets must give a first approximation to a set combining the symmetric properties of both sets. This expectation is confirmed by the proximity of

$$d_c = (\frac{1}{2})(\ln 4/\ln 3 + \ln 8/\ln 3) \cong 1.577324$$

and the dimension of the Sierpinski gasket, which is $d_c^{(2)} = 1.58496$ as given in Table 1.

Using similar geometrical arguments but this time in the three-dimensional space, we can approximate $d_c^{(3)} = 2.51210$ of Table 1 by a generalized Cartesian product of a Menger sponge (d_1), a three-dimensional Cartesian product of three triadic sets (d_2) and a cube made of 27 small cubes from which the core is iteratively removed (d_3). This way one finds

$$\begin{aligned} d_c &= (\frac{1}{3})(d_1 + d_2 + d_3) = \ln 20/\ln 27 + \ln 2/\ln 27 + \ln 26/\ln 27 \\ &= 2.5284. \end{aligned}$$

Again this is quite close to $d_c^{(3)} \cong 2.51210$.

7. POSSIBLE IMPLICATIONS FOR TURBULENCE

It is generally difficult to understand how manifestly real quantities such as a critical value of a certain process could be determined universally and without detailed knowledge of the system. Nevertheless, in the present case the critical value is a Hausdorff dimension which is a rather global property of the system, defined primarily to describe morphological and rather general characteristics of the set. In view of the many found universalities and in particular the Ruelle–Takens–Newhouse scenario of turbulence* [9–11], it would not be entirely surprising to be able to obtain a critical space or dimension of a general nature like the one at hand ($d_c^{(4)} \cong 4$ and $d_c^{(5)} \cong 6.3$). It is also clear that the model presented here avoids an inherent difficulty in the analysis of continuous problems in higher dimensions, such as fully developed turbulence. The analysis of the simplest problems of this kind leads

*The theorem of Sarkovskii might be a good example for an abstract ordering of integers with direct connection to dynamical systems.

to nonlinear partial differential equations. This is ‘infinitely’ more difficult to solve than any comparable nonlinear, but ordinary differential equations. However, since we are anyway chiefly interested in global properties, it is a fortunate situation that we can proceed directly to these quantities without solving the governing partial differential equation.

It must be admitted however that the precise physical relevance of the preceding mathematics is debatable. Nevertheless, there are already many published experimental results which could lead us to believe that $d_c^{(4)}$ and $d_c^{(5)}$ may indeed have a physical significance. We discuss here three of them.

First there is the work of Duong-van [12] concerning a reaction–diffusion model where he found the fractal dimension to be 2.6 in the oscillatory instability, but 6.0 in the turbulence regime. Second there are the experimental results of Paidoussis *et al.* [13] who investigated the vibrations of a mechanical model which represents a fluid–structure interaction device. Using Ruelle–Packard method they found that the dimension of the attractor of the set in the chaotic regime is about $d \cong 6.0$. Noting that the phase space of their model is reported to be $d = 4$ (we think it might be modeled better by $d = 5$) we feel that this is an indication for the relevance of $d_c^{(5)} \cong 6.3$. Finally we may mention that Held *et al.* [14] found for the chaotic behaviour of helical instabilities in an electron–hole plasma, that the measured fractal dimension for a spacially coherent and temporally chaotic helical density waves is $d \cong 3$ while beyond the onset of spacial incoherence the fractal dimension becomes indeterminately large $d > 8$. This indicates to us that beyond a phase space of $d_E = 4$, the system becomes Anosov-like and highly unstable as predicted by our elementary analysis. However we stress that only very accurate measurements on highly controlled experiments can give reliable information. In fact, the Ruelle–Packard method of reconstructing the dynamics of a system is far too inaccurate particularly when based on actual experimental signals. It is also important to realize that turbulence in the present sense does not mean a high dimensional strange attractor of a dissipative system, but an ergodic almost Hamiltonian type of deterministic chaos.

Another important observation is the resilience of the ergodic limit $d_c^{(4)}$ towards the perturbative influence of different coexisting fatter fractals which have an ergodic limit lower than $d_c^{(4)}$. In fact, we can easily show this using the following averaging arguments. Consider a dynamic for which the first four ergodic dimensions are equi-probable. Remembering that

$$d_s^{(n)} = \sqrt[n-1]{n}$$

we find that the average escalation factor is

$$\tilde{d}_s = \sum_{i=0}^{i=n} (i\sqrt[i-1]{i})/n.$$

For $n = 4$ we thus have

$$\begin{aligned} \tilde{d}_s &= (1 + 2 + \sqrt{3} + \sqrt[3]{4})/4 \\ &\cong 1.57989. \end{aligned}$$

The corresponding critical values are

$$\tilde{d}_c^{(4)} \cong (1.57986)^3 = 3.94327$$

and

$$\tilde{d}_c^{(5)} \cong 6.229.$$

These are very close to those corresponding to a single triadic Cantor set. The ergodic limit $d_c^{(4)}$ is indeed resilient toward disturbance by other Cantor sets (see Appendix 2).

The second observation is with regard to the occupational number \bar{n} which gives the number of non-empty cells. Clearly \bar{n} decides on the side length ε of the minimum hyper-lattice which is used to cover our n dimensional points set in the sense of Hausdorff. Consequently we have

$$\varepsilon = \sqrt[n]{\bar{n}}.$$

Evaluating this for $n = 2$ to $n = 8$ we find that $1.7 < \varepsilon < 1.87$. Since $\ln \varepsilon \cong \ln 2$ we are justified in saying that our entropy in bits may be interpreted as an information dimension as we have done earlier by setting $S_I^{(n)}/K_B \cong I$. In fact \bar{n} has an interesting relation to n , namely $\bar{n} < n^2$ for $n < 6$ and $\bar{n} > n^2$ for $n > 6$ showing again abrupt qualitative behaviour at a critical dimension.

Finally we would like to point out a cyclic rotation between chaotic and ergodic behaviour as dimensionality is increased if we introduce the following elementary renorming step. For that we take first the absolute value of the codimension $|c| = |d_c^{(n)} - d_c^{(n)}|$ then by subtracting from $|c|$ all multiples of n that means keeping only 'non-ergodic' parts of the dynamic, we obtain the renormed absolute value of the codimension $|\bar{c}|$. If we then plot the relative value of $|\bar{c}|$, namely $|\bar{c}|/n$ against n , we observe an interesting waves of deterministic chaos followed by ergodic chaos. Incidentally, the maximum value $|\bar{c}|/n$ corresponds exactly to $n = 6$ which is the same dimension predicted by the criterion $\bar{n} \leq n^2$ for $n \leq 6$. Since ergodic chaos similar to order may be viewed as a case of minimum complexity, we see in this result a hint at the possibility of stabilizing chaotic behaviour by injecting more chaotic behaviour into the system.

8. SPECULATIVE IMPLICATIONS FOR QUANTUM MECHANICS—BOSCOVICH COVARIANCE

Following for instance Hawking in accepting the idea of a spacialized time on the extremely small micro scale, we see that our analytical as well as statistical results will remain applicable and may be given analogous interpretation in real space. One could argue as done for instance recently [20] that real space-time on the quantum scale is a Cantorian space which represents a four dimensional analogue of a Peano–Hilbert curve. The near equality of $d_c^{(4)}$ and $d_E^{(4)}$ in this case insures a smooth transition from the dimensionality of micro to the dimensionality of the macro space-time. We will thus imagine this space-time to be inhabited by imaginary particles, which we will refer to as Cantorions, a name motivated by similar reasoning to Kruskal's solitons. It is easy to see that the ergodicity of $d_c^{(4)}$ implies that the two dimensional picture of a Cantorion particle moving on a $d_c^{(4)}$ Cantorion geodesics resemble a "Poincaré" section in real space-time with a Hausdorff dimension $D_c \approx 2$. This conclusion may be supported by two experimental observations. The first noticed by Feynman *et al.* [19] is that the paths of a quantum particle are non-differentiable curves. The second is the well known fact that the fractal dimension of the Brownian motion, a classical analogue of the motion of an electron, is $D = 2$.

Now we should remember that our results state that $d_c^{(4)}$ is almost Peano-like ($D \approx 2$) but not exactly a Peano curve ($d_c^{(4)} = 4$, $D_c \approx 2$). There is here an important difference. In the first case ($D \approx 2$) the length of the path of a Cantorion is infinite while in the second case ($D \approx 2$) the "length" is zero. In both cases however we would have a physically unacceptable infinite velocity.* But we have to be careful here. First our $d_c^{(4)}$ space-time has

*This may be relevant to the interpretation of Aspect's experimental results of Bell's inequality and the Einstein–Podolsky–Rosen and Bohm gedanken experiment.

the paradoxical property of being four dimensional but because of probability $\Omega^{(4)} = (\ln 2/\ln 3)^4$, 84% of its volume is “empty” empty space (that mean nothingness). We have no way of visualizing this, let alone testing it experimentally. Further more our classical definition of velocity (dv/dt) is no more applicable since space as well as time are non-differentiable. There might be an easy way out however by postulating for instance a smallest time and a smallest length as suggested for instance by Heisenberg many years ago. Non-differentiability could be treated somehow using an outgrowth of mathematical logic “which unchained the canard” [26], namely non-standard analysis, for instance [27]. We have no clear idea about a final precise formulation but these are all clearly important points for future research efforts. Regardless of all of this we must of course overcome some physico-logical difficulties connected with the nature of “nothingness”. This old philosophical problem was centre to the Greek philosophy as it is at the heart of modern existentialism (Heideger, Sartre etc.). In its simplest form, the paradox is that a discontinuous space separated by nothing is another way of saying that it is not separated at all and consequently continuous. Nevertheless by introducing the Hausdorff length in place of the ordinary length, we see clearly how Cantorian space dependency on the resolution of our measurement bears a striking similarity to Heisenberg’s uncertainty principles, particularly in the Feynman formulation given in [19].

Our present position may be stated as follows: We are most definitely sympathetic to some of the criticism of the Copenhagen interpretation as expressed by many starting from Einstein up to Bell. However we are also sympathetic to Bohr and Heisenberg’s views. Einstein’s god may or may not be playing dice, but it could not be up to us to decide that for him. The present Cantorian picture for instance, geometrical and classical at heart as it may be, only transfers the problem to another place. Metaphysics in the modern general meaning lays deep at the roots of transfinite sets and its geometrical applications to structures such as Peano–Hilbert curves and Cantor sets, a fact of which Cantor himself was well aware. Cantor was however full of doubts about the adequacy of Newton’s “materialistic” view of the world, essentially the only physics of his time and he has made more than one statement to this effect. He said for instance about his ideas [21]: “They aim at a more exact, more complete, finer knowledge of nature itself than can be achieved through Newtonian principles”.

It is no doubt the great achievement of Mandelbrot [24] to have rediscovered Cantor’s work for physics. To underplay this, because Cantor’s work was in any case there for physicists to pick up, is like giving Columbus no credit for discovering America because it was always there or because some Viking lost in the ocean found America first.

Although Cantor’s views are surely overstated we are convinced that it is “a paradise from which quantum mechanics should not be deprived” to paraphrase a famous sentence of Hilbert. In conclusion we would like to point out a possibly important connection to the work of an eighteenth century avant-garde of relativity, Boscovich [22], who has suggested in paragraph 2 of his article what we may term today “scale covariance”. An extremely interesting discussion of Boscovich covariance in connection with quantum mechanics was given recently by Roessler [23]. He did not however concentrate on the point of scale covariance and did not suggest the way of including it mathematically in general relativity. Nor did he point out the connection to a space-time structure with this kind of “most general” covariance. We are also not yet in a position to indicate how Boscovich covariance requirements could be met in the simplest and most general way analytically. Our guess however is that it might not be all that difficult. It is in fact one more direction for future research effort with the goal of building a general theory in which: “. . . we will not feed time in any deep reaching account. We must drive time and time only in the continuum idealization out of it. Like wise with space”, as declared by Wheeler [25] a man

who seems to have anticipated, probably before anyone else (with the possible exception of Menger), the “nonclassical” structure of micro space-time.*

CONCLUSION

Using set theoretical and statistical methods we attempted to elucidate certain universal behaviour connected to the Hausdorff dimension of higher order triadic Cantor sets. The results seem to confirm the conjecture that a four-dimensional phase space constitutes a topologically critical situation and that the associated dynamics are typically space filling. As expected the result of the modified Gibbs–Shannon entropy model is quite accurate compared to $d_c^{(n)}$ only when $n > 3$. It is almost exact for $n = 8$. By contrast the result of the modified Maxwell–Boltzmann model is quite reasonable for $n < 6$ and deteriorates thereafter. Both results as well as that of the Bose–Einstein statistics reinforce qualitatively our conjecture that $n = d_c^{(n)} = 4$ constitutes a quasi-ergodic state and that for $n = 5$ the semiloops of the basic Sierpinski–Peano set (curve) becomes homoclinic loops [17] leading to a “phase space” made entirely of non-wandering saddle points. A possible paradigm for this dynamics is the ‘S’ shaped twisted and bent (like a Cartan space with torsional tensor) horse shoe with a cardinality $\bar{S} = 3$ as discussed for instance in [15]. It is important to note that our use of an effective geometric entropy which sums over all configurations as well as the different arrangements of the particles is appropriate for the geometrical form of a nonperiodic (strange) behaviour of a dynamic system which although confined in space, never really reaches a steady state in the classical sense. The concept internal time is hidden in a discrete and finite form in this summation. It is also important to realize that the additive property of some total complexions may be appropriate for the description of a deterministic but chaotic set. Since the statistical view makes no assumptions about the nature of the organizing Cantor set, we are justified in concluding that the triadic set represents the most probable back bone Cantor set of a typical strange behaviour and might indeed be fundamental to dynamical systems as implied in earlier work [1, 2, 15]. In fact the results for $n = 4$ leads us to conjecture that Anosov diffeomorphism which is believed to be rare in general [16] might be typical for four dimensions and thus not rare at all in this topologically critical dimension.

The idea of reducing complexity by intentionally driving the system towards an ergodic state by injecting more ‘chaos’ into it might be quite important in nonlinear control theory and warrants further careful and precise considerations.

Finally, the relevance of the preceding results for quantum mechanical real space-time, as well as some connections to Feynman path integral method, non-standard analysis and Boscovich covariance, are discussed.

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*One could argue that Wheeler’s foam-like space-time at the Planck length scale is related to our Peano-like space-time at the Compton and the de Broglie length scale.

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APPENDIX 1

Calculation of a weighted ensemble \tilde{W}_t

It is perhaps of interest to show how to drive an ensemble for which $\delta_B^{(4)}$ is exactly equal to the phase space dimension and $d_t^{(4)}$. The assumption which we make about the different probabilities for the $\delta_B^{(n)}$ entropy is that they are proportional to the relative complexion. Consequently the weighted total number of complexion $\tilde{W}_t^{(n)}$ can be written as

$$\tilde{W}_t^{(n)} = (1/\bar{W}_t^{(n)}) \sum_{i=1}^{P(n)} (\bar{W}_i^{(n)})^2$$

where \bar{W}_i/\bar{W}_t is the probability of the occurrence of a partition \bar{W}_i and \bar{W}_t is the total number of partitions. For $n = 4$ for which $P(n) = 5$ and $\bar{W}_t = 47$ we have thus

$$\tilde{W}_t^{(4)} = \frac{1}{47} [(24)^2 + (12)^2 + (6)^2 + (4)^2 + (1)^2] = 16.4468.$$

Since \tilde{W}_t must be integer, the entropy is given by

$$\tilde{\delta}_B^{(4)} = \frac{1}{\ln 2} \tilde{S}_B^{(4)}/K_B = \ln 16/\ln 2 = 4.$$

The results for $n = 1$ to $n = 8$ are displayed in Table 2. One could of course use the weighted entropy and arrive in an analogous way to the following formula

$$\tilde{\delta}_B^{(n)} = \tilde{S}_B^{(n)}/K_B = \frac{1}{\ln 2} \left(\frac{1}{\sum_{i=1}^{P(n)} \bar{W}_i^{(n)}} \right) \left[\sum_{i=1}^{P(n)} \bar{W}_i^{(n)} (\ln \bar{W}_i^{(n)}) \right].$$

Applying this in case of $n = 4$, one finds

$$\begin{aligned} \tilde{\delta}_B^{(4)} &= \frac{1}{\ln 2} \left(\frac{1}{47} \right) [24 \ln 24 + 12 \ln 12 + 6 \ln 6 + 4 \ln 4 + 1 \ln 1] \\ &= 3.75677. \end{aligned}$$

The rest of the numerical results are in Table 2. Note that the difference between the two methods is numerically marginal and qualitatively identical.

Table 2

n	$\log_2 \tilde{\eta}_t$	$\ln(n)^n$	$\log_2 \tilde{\eta}_t = \tilde{S}/K_B$	$\tilde{S}^{(n)}/K_B$	S_I	$d_c^{(n)}$
2	1.58496	1.386	1	0.666	0.5	1.58496
3	3.3219	3.295	2.3219	2.026	1.3899	2.5121
4	5.554	5.54	<u>4.0</u>	3.756	3.18872	3.98159
5	7.9425	8.04	<u>6.303</u>	5.9419	5.537	6.31067
6	10.645	10.75	8.7879	8.3114	10.0836	10.00218
7	13.4869	13.62	11.56319	11.025	15.5169	15.853
8	16.545	16.63	14.5285	13.937	25.35	25.126

Numerical estimation of $\tilde{\eta}_t$

Calculation of the complex ion $\tilde{\eta}_t$ is a tedious operation for $n > 5$. However a good approximation for $\tilde{\eta}_t$ can be obtained using the following procedure. First one notices the proximity of the numerical values of columns one and two of Table 2. Consequently

$$\tilde{\eta}_t^{(n)} \cong e^{n \ln n \ln 2}.$$

For $n = 4$ one finds $\tilde{\eta}_t^{(4)} = 46.694 \cong 47$ which is the exact value.

APPENDIX 2

An alternative derivation and additional remarks

Lacking a direct geometrical measure, we use indirectly the Hausdorff "length" to define a probability quotient Ω analogous to geometrical probability

$$\Omega = d(\text{sub set})/d(\text{set})$$

For a Cantor set living in one dimension, the probability of finding a Cantor point is thus

$$\Omega^{(1)} = d_c/d_c^{(1)}$$

where d_c is the Hausdorff dimension, $d_c^{(1)} = d_c/d_c = d_E = 1$ and E means Euclidean. Consequently finding a Cantor point (a Cantorion) in n dimensions can be found using the multiplications theorem of independent events in a probability space [3, 4]

$$\Omega^{(n)} = (d_c/d_c^{(1)})^n = (d_c)^n.$$

On the other hand we have

$$\Omega^{(n)} = d_c/d_c^{(n)}.$$

Therefore one finds

$$d_c^{(n)} = (1/d_c)^{n-1}.$$

It is important to notice that if d_c is assigned the dimension zero ($d_c^{(0)}$) then we could use a modified Menger-Urysohn dimensional system ($d_M^{(n)}$). That way our simple formula may be interpreted as describing a bijection between $d_M^{(n)} = n$ and $d_c^{(n)}$. The new dimension d_c is now clearly unidirectional with an absolutely empty space (d_c^0) corresponding to $n = -\infty$ as an initial singularity. We have therefore two distinct regions. The first, a unit interval ($d_c^0 - d_c^{(1)}$) with infinite numbers of virtual sets corresponding to the negative region $d_M^{(1)} - d_M^{(-\infty)}$.

The dimensions of the real sets are then given by a primitive transformation, namely inversion, of the dimension of the virtual sets. This way we find the second region of $d_c^{(n)}$ which corresponds to $d_M^{(1)} - d_M^{(+\infty)}$.

Seen this way our choice of $d_c^{(0)} = \log 2/\log 3$ is justified *a posteriori* by its proximity to $\phi = (\sqrt{5} - 1)/2$ because the golden mean represents "in some vague sense" an average of all "possible" backbone Cantor sets laying in the unit interval $(0 - 1)$ of the virtual sets. This is some what reminiscent of Feynman's path integral formulation [19]. In the same time $d_c^{(0)} = \log 2/\log 3$ is of course the simplest possible Cantor set which allows for replication by doubling. Having said that it must be clearly stated that the simplest initial assumption "for a world set" does not necessarily lead to the simplest next structure in higher dimensions. It is however quite reasonable to assume that simplicity in nature must mean the simplest initial choices regardless of any "unknown" future evolution. Our $d_c^{(n)}$ is thus an optimum choice from more than one point of view.

The relation between the dimensionality ($d_c^{(n)}$) and the randomness and chaoticness of the corresponding set is also of interest in our previous context. The Cantor set $d_c^{(0)}$ is clearly ordered and so is $d_c^{(1)} = 1$ and $d_c^{(2)} = \log 3/\log 2$ corresponding to a line and Sierpinski gasket respectively. By contrast there seems to be no clearly ordered geometrical structure leading to a points set corresponding to $d_c^{(3)} = (\log 3/\log 2)^2$. Our guess is that it is a random as well as a chaotic set. Finally $d_c^{(4)} \cong 4$ is, as mentioned earlier, ergodic and thus nonchaotic. All these intuitive conclusions seem to agree with well known criteria for the dimensionality of the phase space of chaotic flow.